Robust Stabilization of Input/Output Linearizable Systems under Uncertainty and Disturbances

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In recent years, differential geometric techniques have been used to transform nonlinear systems into linear systems. Once such equivalent linear systems are obtained, classical linear controllers are designed to achieve desired stability and performance properties. A major criticism against these techniques is their lack of guarantee for robustness. In particular, the design of controllers for transformed nonlinear systems under the influence of both disturbances and (parametric) modeling errors is not well-known. This article presents a methodology to design robust stabilizing controllers for such uncertain and perturbed nonlinear systems.

For feedback linearizable systems, the method guarantees that the nonlinear system has nominally linear input/output dynamics and is stable for the given class of bounded parametric uncertainty and disturbances. The new concepts and the proposed design procedure are shown for an isothermal reactor with second-order kinetics.

Introduction

In the last decade, the control literature has witnessed a surge of differential geometric techniques for the control of nonlinear systems. In one form or another, the proposed approaches are based on transforming nonlinear systems into linear systems. The required transformations include state coordinate changes and/or input transformation with nonlinear state feedback. Whenever applicable, the method has the potential to alleviate some of the control problems introduced by approximation errors involved in local linearization of strongly nonlinear systems. The review articles by Kravaris and Kantor (1990a,b) and Kravaris and Arkun (1991) give a survey of the geometric methods, discuss their relevance to process control, and cite several chemical engineering applications.

Despite the significant progress and interest in nonlinear geometric control, the transformation techniques rarely address nonlinear systems with modeling uncertainties and unmeasured external disturbances which are so predominant in process control. Robust stabilization of feedback linearizable nonlinear systems without considering outputs against dynamic uncertainty has been addressed by various techniques including: variable structure control by Sira-Ramirez (1986); second method of Lyapunov (Spong and Sira-Ramirez, 1986); stable factorization approach by Spong and Vidyasagar (1985a,b); and adaptive control by Taylor et al. (1988). All these works assume that discrepancy between the model and the plant satisfies certain "matching conditions." Similarly, the design of robust controllers for systems transformed in the input/output sense has been investigated with similar matching conditions (Kravaris and Palanki, 1988).

In this article we study robust stabilization under bounded *parametric* uncertainty and disturbances. The "matching conditions" are required to be satisfied only partially. The problem addressed can be stated as follows:

> Consider a nonlinear system which is feedback-transformable to a linear system in the input/output sense under nominal conditions. Now assume that this nonlinear system is perturbed by bounded disturbances and is subject to parametric modeling uncertainty. Given bounds on the disturbances and parametric uncertainty, how can one design controllers which will guarantee that

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the transformed nonlinear system has nominally linear input/output dynamics and is stable for the given class of disturbances and parametric uncertainty?

The article discusses the necessary mathematical preliminaries from differential geometry, the class of nonlinear systems, and the uncertainty description used in this work. The structure is developed for the uncertain nonlinear system under nominal feedback linearization. One class of robust controllers based on the second method of Lyapunov is synthesized, and the controller gain is calculated from different types of algebraic Riccati equations depending on whether or not matching conditions are satisfied. A reactor example illustrates an application of the proposed technique.

Preliminaries

The necessary background is given here; for details, the reader is referred to differential geometry books (Boothby, 1975; Casti, 1985; Isidori, 1989) or the review articles by Kravaris and Kantor (1990a,b).

Lie derivative

Let us consider a C^∞ (infinitely differentiable) real-valued function h and a vector field f on R^n. The Lie derivative of the function h with respect to the vector field f is defined as:

$$L_f h = \langle dh, f \rangle = \sum_{i=1}^{n} \left(\frac{\partial h}{\partial x_i} \right) f_i$$

where dh is the gradient of h (from now on, we will represent the gradient of any scalar function by a *row vector*). Note that $L_f h$ is also a real-valued function. Therefore, we can define higher Lie derivatives of h with respect to the vector field f by:

$$L_f^0 h = h$$

$$L_f^1 h = \langle dh, f \rangle$$

$$L_f^2 h = L_f L_f h = \langle dL_f h, f \rangle$$

$$L_f^k h = L_f L_f^{k-1} h = \langle dL_f^{k-1} h, f \rangle$$

If we introduce another C^∞ vector field g, one can define the Lie derivative of h with respect to two different vector fields by:

$$L_g L_f^k h = \langle dL_f^k h, g \rangle \quad k = 0,1,2,\ldots$$

Input/output linearization (Kravaris and Chung, 1987)

Consider the nonlinear system (NLS) of the form:

$$\dot{x} = f(x) + g(x)u$$
$$y = h(x) \tag{1}$$

The NLS is said to be I/O-linearizable, if there exists a nonlinear state feedback input transformation which renders the closed-loop dynamics between the output (y) and input (v) linear. I/O linearization is feasible as long as the NLS has finite relative order, where the relative order is defined as the smallest integer for which $L_g L_f h^{r-1}(x) \neq 0$.

Partial linearization (Byrnes and Isidori, 1985; Calvet, 1989)

Assume that the NLS is I/O-linearizable with relative order r. Then, there exists a state and input transformation with feedback (not unique):

$$z = T(x)$$
$$u = S(x, v)$$

transforming the NLS to a partially linearized system where the output is influenced only by the linear, controllable part of the system:

$$\dot{z}^{(1)} = Az^{(1)} + bv$$
$$\dot{z}^{(2)} = \xi(z^{(1)}, z^{(2)})$$
$$y = cz^{(1)} \tag{2}$$

with $z = [z^{(1)}, z^{(2)}]^T$, $z^{(1)} \in R^r$ and $z^{(2)} \in R^{n-r}$. The pair (A, b) is in the Brunovski Canonical Form (BCF):

$$A = \begin{bmatrix} 0 & 1 & 0 & . & 0 \\ 0 & 0 & 1 & . & 0 \\ . & . & . & . & . \\ 0 & 0 & 0 & . & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \quad b = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

and $c = [1\ 0\ 0 \ldots 0]$.

The transformations are given by:

$$z_i = T_i(x) = L_f^{i-1} h(x) \qquad i = 1, \ldots, r$$
$$z_j = T_j(x) \text{ such that } L_g z_j = 0 \quad j = r+1, \ldots, n \tag{3}$$

$$u = -\frac{L_f^r h(x)}{L_g L_f^{r-1} h(x)} + \frac{1}{L_g L_f^{r-1} h(x)} v = \alpha(x) + \beta(x) v$$

In the sequel, these transformations will be called *nominal* transformations. Note that when $r = n$, the nonlinear state equation $z^{(2)}$ disappears and the NLS (Eq. 2) becomes *totally linear*:

$$\dot{z} = Az + bv$$
$$y = z_1 \tag{4}$$

This special case is called the Su-Hunt-Meyer linearization (Hunt et al., 1983) with linear I/O dynamics. Both the complete state space and the I/O dynamics are linearized due to $r = n$.

Zero dynamics (Isidori, 1989)

The zero dynamics of the NLS is defined by the nonlinear part of the partially linearized system (Eq. 2):

$$\dot{z}^{(2)} = \xi(z^{(1)}, z^{(2)})$$

where

$$z^{(1)} = \left[y, \frac{dy}{dt}, \ldots, \frac{d^{r-1}y}{dt^{r-1}} \right]$$

is considered as the input to the zero dynamics. The NLS is called minimum phase if and only if the unforced zero dynamics, $\dot{z}^{(2)} = \xi(0, z^{(2)})$, has an asymptotically stable equilibrium point.

I/O Linearization of Uncertain Nonlinear Systems

We consider the class of uncertain NLS given by:

$$\begin{aligned} \dot{x} &= f(x,p) + g(x,p)u + \chi(x,p,d) \\ y &= h(x) \end{aligned} \tag{5}$$

where $f(x, p)$, $g(x, p)$, and $\chi(x, p, d)$ are smooth vector fields on $\boldsymbol{R}^n$; $h(x)$ is a smooth real vector valued function. The vector of uncertain parameters, p, belongs to a closed bounded set $\Omega_p \subset R^l$. Similarly, the vector of unmeasured external disturbances, d, belongs to a closed bounded set $\Omega_d \subset \boldsymbol{R}^m$. Uncertain parameters and external disturbances are assumed to be time-invariant and time-varying (but bounded), respectively. Under nominal conditions χ is such that $\chi(x, p^0, 0) = 0$. It is also required that the following regularity conditions are satisfied.

Regularity conditions (Calvet, 1989)

a. The set $\{L_f^{i-1}h(x) i = 1, \ldots, r\}$ is a linearly-independent set for all p in Ω_p.

b. The states $z^{(2)}$ are linearly-independent and satisfy $L_g z_j = 0$ for $j = r+1, \ldots, n$ for all p in Ω_p.

These regularity conditions are needed to map the NLS to a quasi-linear system, which is to be defined below. It can be shown that there always exists a neighborhood Ω_p containing the nominal values p^0 for which the above regularity conditions are satisfied. The size of this neighborhood depends on the particular problem and type of uncertainty studied.

Since the true parameter values are unknown, the transformations will be computed for the nominal values $p = p^0$ and applied to the uncertain NLS (Eq. 5). In the following, we develop the structure of such transformed uncertain NLSs.

Structure of the transformed NLS

Let δf and δg denote the parametric vector field mismatches due to uncertain parameters:

$$\begin{aligned} \delta f &= f(x,p) - f(x,p^0) \equiv f - \hat{f} \\ \delta g &= g(x,p) - g(x,p^0) \equiv g - \hat{g} \end{aligned}$$

Next certain matching conditions are defined with respect to the vector field mismatch δg.

First Matching Conditions. We say that the vector field δg satisfies the matching conditions if:

$$\exists \text{ a scalar } E(x,p) \equiv E \text{ s.t } \delta g = E\hat{g}$$

This very first matching condition implies that the uncertainty vector field δg lies in the span of the nominal vector field $\hat{g}$. A second matching condition with respect to δf will be defined later. It, however, will be shown that this second matching condition can be relaxed.

With the regularity conditions and the first matching condition, applying the *nominal* transformations (Eq. 3) to the uncertain NLS results in the following generic structure:

$$\begin{aligned} \dot{z}^{(1)} &= Az^{(1)} + b\{v + \eta[z, \delta g(z;p), v] + L_{(\delta f + \chi)} z^{(1)}\} \\ \dot{z}^{(2)} &= \hat{\xi}(z^{(1)}, z^{(2)}) + L_{(\delta f + \chi)} z^{(2)} \\ y &= cz^{(1)} = z_1 \end{aligned} \tag{6}$$

which contains the following terms induced by uncertainty:

$$\begin{aligned} L_{(\delta f + \chi)} z^{(1)} &= [L_{(\delta f + \chi)} z_1, \ldots, L_{(\delta f + \chi)} z_r]^T \\ L_{(\delta f + \chi)} z^{(2)} &= [L_{(\delta f + \chi)} z_{r+1}, \ldots, L_{(\delta f + \chi)} z_n]^T \\ \eta[z, \delta g(z;p), v] &= L_{\delta g} z_r \hat{\beta} v + L_{\delta g} z_r \hat{\alpha} \end{aligned} \tag{7}$$

The derivation is given in the Appendix. Several important observations about Eqs. 6 and 7 are in order. First note that both the input v and uncertainty due to δg "enter" the NLS through the linear operator b. This important property is due to the first matching condition. The first state equation for $z^{(1)}$ is quasi-linear: it has the nominal linear part (A, b) plus new nonlinear perturbations η and $L_{(\delta f + \chi)} z^{(1)}$ induced by uncertainty and disturbances. The second state equation for $z^{(2)}$ contains the nominal nonlinear zero dynamics plus new nonlinearities due to uncertainty and disturbances. When there is no uncertainty ($\delta f = \delta g = 0$) and no disturbances ($\chi = 0$), the NLS (Eq. 6) reduces to the nominal partially linearized system (Eq. 2) with linear I/O dynamics as expected.

When the relative order is equal to the dimension of the system, the zero dynamics vanish and one has the following special structure (Calvet, 1989):

$$\begin{aligned} \dot{z} &= Az + b\{v + \eta[z, \delta g(z;p), v] + L_{(\delta f + \chi)} z\} \\ y &= z_1 \end{aligned} \tag{8}$$

and $\eta[z, \delta g(z;p), v] = L_{\delta g} z_n \hat{\beta} v + L_{\delta g} z_n \hat{\alpha}$.

Stabilization of these types of nonlinear systems without zero dynamics is addressed in Calvet and Arkun (1989). Finally note that without uncertainty and disturbances, the NLS (Eq. 8) is equivalent to the totally linearized nominal system (Eq. 4), as expected.

The goal of this article is to design stabilizing controllers that guarantee ultimate stability of the transformed uncertain NLS (Eq. 6). This will automatically imply stability for the original uncertain NLS (Eq. 5) due to the diffeomorphism or one-to-one mapping between the original and transformed states, x and z, respectively. It is well-known that if the nominal NLS (Eq. 2) has stable zero dynamics, its stabilization is easily achieved (locally) under a linear state feedback control law:

$$v = -[k_1, \ldots, k_r] z^{(1)} \tag{9}$$

There, however, is no guarantee that this controller will stabilize the transformed uncertain NLS (Eq. 6). In the next section we give the controller synthesis that achieves this. Note that in the sequel we address state feedback only. The effects of uncertain models in the observer that is required in the case of robust stabilization by output feedback is not considered.

Controller Synthesis for Robust Stability

In the previous section, we saw that the structure of uncertain NLSs after nominal transformations consist of two subsystems:

Quasi-linear part:

$$\dot{z}^{(1)} = Az^{(1)} + b\{v + \eta[z,\delta g(z;p),v]\} + L_{(\delta f+\chi)}z^{(1)} \quad z^{(1)} \in \boldsymbol{R}^r \tag{10}$$

Perturbed zero dynamics part:

$$\dot{z}^{(2)} = \xi(z^{(1)},z^{(2)}) + L_{(\delta f+\chi)}z^{(2)} \quad z^{(2)} \in \boldsymbol{R}^{n-r}$$

In this section, synthesis of stabilizing controllers for the above uncertain NLS is presented. The basic idea is to apply δ-stabilization technique proposed in Calvet and Arkun (1989) to the quasi-linear part of the NLS. This, together with the boundedness of the perturbed zero dynamics, will guarantee the stability of the overall system. Before giving the controller synthesis, several definitions are needed. The first two are adopted from the literature and stated in a form that applies to our problem.

Ultimate boundedness (Corless and Leitman, 1981)

A solution $z(\cdot):[t_0,\infty] \rightarrow \boldsymbol{R}^n$ with initial conditions $z(t_0)=z_0$ of the system of Eq. 6 is said to be ultimately bounded with respect to a closed set $B(\delta)=\{z \in \boldsymbol{R}^n;\ \|z\| \le \delta\}$, if there exists a nonnegative constant (time) $T(z_0,\delta)<\infty$ possibly dependent on z_0 and δ but not on t_0 such that $z(t) \in B(\delta)$ for all $t \ge t_0 + T(z_0,\delta)$. We call $B(\delta)$ a ball of radius δ.

δ stability (Schmitendorf, 1988a,b)

The nonlinear system (Eq. 6) is said to be δ-stabilizable if there exists a linear state feedback control law $v_\delta = -K_\delta z^{(1)}$ such that for all admissible modeling parameters in Ω_p, disturbances in Ω_d and initial conditions in a ball $B(r_o)$, the solution $z(t)$ is ultimately bounded in $B(\delta)$. The control v_δ is then called a δ-stabilizing control.

Perturbed zero dynamics (Calvet, 1989)

1. The (forced) perturbed zero dynamics of the NLS (Eq. 6) are defined by the dynamics of the state $z^{(2)}$:

$$\begin{aligned}\dot{z}_{r+1} &= \xi_1(U_1,\ldots,U_r;z^{(2)}) + L_{(\delta f+\chi)}z_{r+1}\\ &\ \vdots\\ \dot{z}_n &= \xi_{n-r}(U_1,\ldots,U_r;z^{(2)}) + L_{(\delta f+\chi)}z_n\end{aligned}$$

where $U_1,\ldots,U_r$ are exponentially decaying inputs ultimately bounded by $\overline{U}_i$: $\exists T>0\ s.t.\ |U_i(t)| \le a_i e^{-k_i t} + \overline{U}_i\ a_i>0,\ k_i>0,\ 0<\overline{U}_i<\infty\ i=1,\ldots,r\ t>T$.

Inputs U_i's represent the perturbed states of the quasi-linear system: $z_1,\ldots,z_r$ and $\overline{U}_i$'s are equal to their steady-state values. It is important to realize that the steady state for these states may no longer be the origin due to changes in the values of the uncertain parameters and disturbances. If $\delta f = \chi = 0$, then $\overline{U}_i = 0$ and the perturbed zero dynamics reduce to the classical unforced zero dynamics as defined by Kravaris (1988).

Locally minimum phase

An uncertain NLS (Eq. 5) mapped by nominal transformations to the NLS (Eq. 10) will be called locally minimum-phase $\forall p \in \Omega_p$ and $\forall d \in \Omega_d$, if and only if for any initial condition $z^{(2)}(0)$ in the neighborhood of the origin and exponentially decaying inputs ultimately bounded by $\overline{U}_i$, $z^{(2)}$ is ultimately bounded:

$$\exists T>0\ s.t. \|z^{(2)}(t)\| < \infty \forall t > T$$

where $\|\cdot\|$ is the usual Euclidean norm.

It is possible that by changing the values of the parameters p or disturbances d, a nominally minimum-phase NLS could become nonminimum-phase. Here, such systems will not be considered. In fact, if a NLS is nominally minimum-phase (nonminimum-phase) for the nominal values p^0, d^0, then it is assumed to remain minimum-phase (nonminimum-phase) for all other values $p \in \Omega_p$ and $d \in \Omega_d$. [It can be shown that if the NLS is minimum phase for p^0 and d^0, then there exists a neighborhood containing p^0, d^0, for which any value in this neighborhood renders the NLS minimum phase as well (Calvet, 1989).] This assumption will be verified in the application of the theory.

Second matching conditions

Now we define the matching conditions for the quasi-linear system (Eq. 10). The parametric uncertainty and disturbance vector field $L_{(\delta f+\chi)}z^{(1)}$ satisfies matching conditions if

$$\exists \text{ a scalar } D(x,p,d) \equiv D\ s.t.\ L_{(\delta f+\chi)}z^{(1)} = bD$$

That is, the uncertainty and disturbance vector field lies in the span of the nominal input vector field b. If the matching condition is *not* satisfied, then $L_{(\delta f+\chi)}z^{(1)}$ is split into vector fields:

$$L_{(\delta f+\chi)}z^{(1)} = (L_{(\delta f+\chi)}z^{(1)})_{um} + (L_{(\delta f+\chi)}z^{(1)})_m$$
$$= \begin{bmatrix} L_{(\delta f+\chi)}z_1 \\ \cdot \\ L_{(\delta f+\chi)}z_{r-1} \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ L_{(\delta f+\chi)}z_r \end{bmatrix}$$

$(L_{(\delta f+\chi)}z^{(1)})_m$ satisfies the matching conditions and will be called matched uncertainty. On the other hand, $(L_{(\delta f+\chi)}z^{(1)})_{um}$ does not satisfy the matching conditions and thus will be called unmatched uncertainty. Note that the other uncertainty vector field due to δg, that is, $\eta[z,\delta g(z;p),v]$ already lies in the span of b due to the first matching condition and is thus matched.

Also define:

$$\eta' = \eta + L_{(\delta f+\chi)} z_r = L_{\delta g} z_r \hat{\beta} v + L_{\delta g} z_r \hat{\alpha} + L_{(\delta f+\chi)} z_r$$
$$\equiv \psi(z,p) v + \phi(z,p,d)$$

This way the quasi-linear part of Eq. 10 can be expressed in its final form for controller design:

$$\dot{z}^{(1)} = A z^{(1)} + b(v+\eta') + (L_{(\delta f+\chi)} z^{(1)})_{um} \tag{11}$$

where nonlinear perturbations due to matched and unmatched uncertainty are shown explicitly. In summary, our method can relax the matching conditions for uncertainty δf and disturbance vector field χ. Uncertainty due to δg, however, has to be matched. Relaxing this matching condition in the context of the present work is still an open problem. Finally, construction of the δ-stabilizing controller requires the following assumptions.

Assumption 1. The norm of $\psi(z,p)$ is bounded in a ball $B(\bar{r})$:

$$\exists \bar{\alpha} \ s.t. \|\psi(z,p)\| \leq \bar{\alpha} \forall z = [z^{(1)}, z^{(2)}] \in B(\bar{r}), \quad \forall p \in \Omega_p$$

Assumption 2. There exists $(\rho, \mu) \in R^2 - \{0,0\}$ both finite such that:

$$\|\phi(z^{(1)}, z^{(2)}, p, d)\|^2 + \|(L_{(\delta f+\chi)} z^{(1)})_{um}\|^2 \leq \rho^2 + \mu^2 \|z^{(1)}\|^2$$
$$\forall z = [z^{(1)}, z^{(2)}] \in B(\bar{r}), \quad \forall p \in \Omega_p, \quad \forall d \in \Omega_d$$

By design the stabilizing controller will guarantee that substate $z^{(1)}$ will be ultimately bounded if assumptions 1 and 2 are satisfied. Assumptions 1 and 2 are always locally satisfied as long as $\|\phi\|$, $\|\psi\|$, and $\|(L_{(\delta f+\chi)} z^{(1)})_{um}\|$ are finite at the nominal operating point ($z=0$, $p=p^0$, $d=0$). There, however, is no guarantee that the zero dynamic states $z^{(2)}$ will remain bounded during the stabilization process so that the conditions in assumptions 1 and 2 ($[z^{(2)}] \in B(\bar{r})$) can be justified. For example, for the nominal case, Sussman (1990) has shown that a globally minimum-phase system may not be globally-stabilizable due to unbounded zero dynamics states. Therefore, an additional assumption regarding the perturbed zero dynamics must be introduced.

Assumption 3. The perturbed zero dynamics are bounded at all time if for any decaying inputs, $z^{(1)}(t)$ ($\|z_i(t)\| \leq a_i e^{-k_i t} + \bar{z}_i$ $i=1,\ldots,r$) and with initial conditions $z^{(2)}(0) \in B(r_0)$, the following holds:

$$\|z^{(2)}(t)\| < \bar{r} \quad \forall t > 0$$

This is indeed the key assumption that allows the robust stabilization of nominally I/O linearized NLSs. Note that in the case $r=n$ this assumption is not needed.

Assumption 3 can be difficult to verify since one needs to solve the differential equations for the perturbed zero dynamics. The following classical theorem (for example, Hale, 1980) is useful in this regard. Basically we will verify assumption 3 through another set of differential equations which are easier to solve.

Theorem 1

Consider two continuous real valued functions $f(z,t,p)$ and $g(z,t,p)$ satisfying the Lipschiptz condition and representing two first-order differential equations:

$$\dot{x} = f[x(t), t, p]$$
$$\dot{w} = g[w(t), t, p]$$

assume that $f[z(t),t,p] \leq g[z(t),t,p]$ for all $z \in R^n$ and $p \in \Omega_p$. Then, if the initial conditions are such that $x(t_0) = w(t_0)$, it implies that $x(t) \leq w(t)$ for all $t \geq t_0$ and all $p \in \Omega_p$.

This theorem is used in the example to verify assumption 3.

Results

The following theorem gives the main δ-stabilization result of the uncertain NLS (Eq. 10).

Theorem 2

Let assumptions 1, 2 and 3 hold. Given δ such that $\bar{r} > \delta > 0$.

Case 1: matched uncertainty, that is, $(L_{\delta f} z^{(1)})_{um} = 0$. If there exists a positive definite matrix P solving the algebraic Riccati equation (ARE):

$$PA + A^T P - Pb[2R^{-1} - (2\bar{\alpha}\|R^{-1}\| + 1)I]b^T P + H/\epsilon^2 = 0$$

for some $\epsilon > 0$ and $H = H^T > 0$, $R = R^T > 0$ satisfying:

$$\delta > \delta^* \equiv \epsilon \sqrt{\frac{\gamma_P \rho^2}{\lambda_*(H) - \mu^2 \epsilon^2}} \text{ with } \lambda_*(H) - \mu^2 \epsilon^2 > 0$$

then the control law:

$$v(t) = -R^{-1} b^T P z^{(1)}(t)$$

is a δ-stabilizing controller for the states $z^{(1)}$ for all initial conditions such that $\|z^{(1)}(t_0)\| < r/\sqrt{\gamma_P}$, $r < \bar{r}$ and $\|z^{(2)}(t)\| < \bar{r}$ for all $t > 0$ due to assumption 3. $\lambda_*(\cdot)$ is the minimum eigenvalue and γ_P is the condition number.

Case 2: unmatched uncertainty. If there exists a positive definite matrix P solving the algebraic Riccati equation (ARE′):

$$PA + A^T P - P\{b[2R^{-1} - (2\bar{\alpha}\|R^{-1}\| + 1)I]b^T - I\}P + H/\epsilon^2 = 0$$

for some $\epsilon > 0$ and $H = H^T > 0$, $R = R^T > 0$ satisfying:

$$\delta > \delta^* \equiv \epsilon \sqrt{\frac{\gamma_P(\rho^2)}{\lambda^*(H) - (\mu^2)\epsilon^2}} \text{ with } \lambda^*(H) - (\mu^2)\epsilon^2 > 0$$

then the control law:

$$v(t) = -R^{-1} b^T P z^{(1)}(t)$$

is a δ-stabilizing controller for the states $z^{(1)}$ for all initial conditions such that $\|z^{(1)}(t_0)\| < r/\sqrt{\gamma_P}$, $r < \bar{r}$ and $\|z^{(2)}(t)\| < \bar{r}$ for all $t > 0$ due to assumption 3.

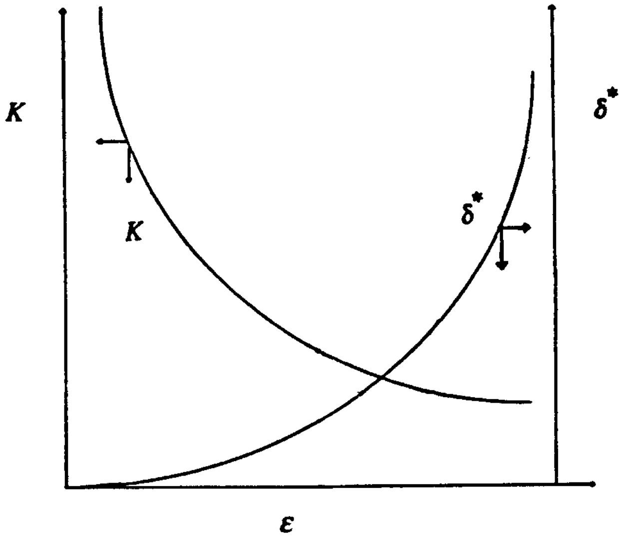

Figure 1a. K and δ^* curves when matching conditions are satisfied.

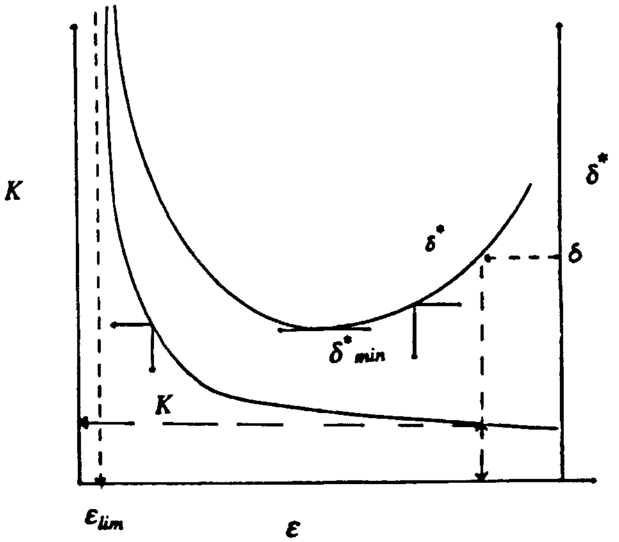

Figure 1b. K and δ^* curves when matching conditions are not satisfied.

Proof

Complete proof is given by Calvet (1989). A sketch of the proof is provided in the Appendix.

One can see that the controller gain $K = R^{-1} b^T P$ does not depend explicitly on the constants ρ and μ introduced in assumptions 2 and 3, rather on the free parameters ϵ, H and R of the algebraic Riccati equation. After solving the Riccati equation for P, the gains are computed as a function of ϵ (parametrized by H and R) and independent of ρ and μ. Similarly, δ^* is also computed as a function of ϵ. The curves $K(\epsilon)$ and $\delta^*(\epsilon)$ have the general shapes in Figure 1. Note that the bounds ρ and μ are needed for the calculation of δ^* only. An algorithm by Calvet and Arkun (1989) shows how to calculate these bounds in an optimal fashion so that conservatism in the values of the controller gains is reduced. The design procedure is as follows (refer to Figure 1b):

1. Plot the gain curve K vs. ϵ with usually $H = I$ and $R = I$.
2. Calculate the best values for ρ and μ according to Calvet and Arkun (1989); plot δ^* vs. ϵ.
3. For a given desired value of δ, pick a δ^* so that $\delta > \delta^*$ (as required by the theorem).
4. Get ϵ from δ^* vs. ϵ curve.
5. Get the stabilizing gains from K vs. ϵ curve.

Remark

If the uncertainty is matched (case 1), there always exists a unique solution $P > 0$ solving the algebraic Riccati equation (ARE) provided that $\{2R^{-1} - (2\alpha \|R^{-1}\| + 1)I\} > 0$. In this case, Figure 1 shows that K goes to infinity and δ^* tends to zero as ϵ goes to zero. Thus, stability can be guaranteed in an arbitrarily small neighborhood of the nominal operating point by increasing the gains. When uncertainty is unmatched (case 2), however, the Riccati equation (ARE′) does not always have a solution. Indeed, one can show that for a given H and R, there exists a value $\epsilon_{\lim}$ below which there is no solution for K. Furthermore, δ^* does not tend to zero and has a lower-bound $\delta^*_{\min}$ which depends on the size of the uncertain parameter set Ω_p and a disturbance set Ω_d (see Figure 1b). Consequently, for a given amount of uncertainty, stability cannot be achieved in an arbitrarily small neighborhood, or given a desired region of stability, δ, the amount of tolerable uncertainty may be large. These are the fundamental limitations introduced by uncertainty that does not satisfy the matching conditions.

Since the theorem is based on a sufficient condition, it will give higher values for the gains than necessary. The least conservative gains can be obtained if ρ and μ are optimally chosen. An algorithm for this is given by Calvet and Arkun (1989).

The design of a stabilizing PI controller can be implemented by simply defining additional state:

$$z'_{r+1} = \int_{t_0}^{t} y(t)dt = \int_{t_0}^{t} z_1(t)dt$$

and augmenting the QLS (Eq. 11):

$$\dot{\bar{z}}^{(1)} = \bar{A}\bar{z}^{(1)} + \bar{b}(v + \eta' + L_{(\delta f+\chi)} z_r) + (\bar{L}_{(\delta f+\chi)} z^{(1)})_{um}$$

$$\bar{A} = \begin{bmatrix} 0 & 1 & . & 0 \\ . & . & . & . \\ 0 & . & 1 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \in R^{(r+1)\times(r+1)} \text{ and } \bar{b} = \begin{bmatrix} 0 \\ . \\ 1 \\ 0 \end{bmatrix} \in R^{r+1}$$

$$(\bar{L}_{(\delta f+\chi)} z^{(1)})_{um} = [L_{(\delta f+\chi)} z_1 \ldots L_{(\delta f+\chi)} z_{r-1} \quad 0 \quad 0]^T$$

The PI δ-stabilizing controller will then be given by:

$$V(t) = -\sum_{i=1}^{i=r} K_i(t) - K_{r+1} \int_{t_0}^{t} y(t)dt$$

The same Riccati equations are solved after replacing (A, b) by $(\bar{A}, \bar{b})$. With this control, the output $y = z_1$ exhibits zero steady-state offset. It can be shown that if the uncertainty is matched, not only the output but all the other states, $z_1, \ldots, z_r$ of the quasi-linear system will exhibit zero steady-state offset (Calvet, 1989).

Table 1. Transformation of NLS and Stability Guarantees with and without Uncertainty/Disturbances

Stabilization guaranteed by feedback (Hunt et al., 1983)

$$\dot{z} = Az + bv$$
$$y = z_1$$

$r = n$

Stabilization guaranteed for minimum phase system (Kravaris, 1988)

$$\dot{z}^{(1)} = Az^{(1)} + bv$$
$$\dot{z}^{(2)} = \xi(z^{(1)}, z^{(2)})$$
$$y = z^{(1)}{}_1$$

$r \neq n$

Without uncertainty and disturbances

$$\dot{x} = f(x,p) + g(x,p)u + \chi(x,p,d)$$
$$y = h(x)$$

With uncertainty and disturbances

$r = n$

δ-stability guaranteed. δ can be arbitrarily small if matching conditions are satisfied (this work)

$$\dot{z} = Az + b(v + \eta(z, \delta g(z;p), v)) + L_{(\delta f + \chi)} z$$
$$y = z_1$$

$r \neq n$

δ-stability guaranteed for perturbed minimum phase systems (this work)

$$\dot{z}^{(1)} = Az^{(1)} + b(v + \eta(z, \delta g(z;p), v)) + L_{(\delta f + \chi)} z^{(1)}$$
$$\dot{z}^{(2)} = \xi(z^{(1)}, z^{(2)}) + L_{(\delta f + \chi)} z^{(1)}$$
$$y = z^{(1)}{}_1$$

In addition to robust stabilization, for the given class of uncertainty and disturbances, robust performance is also guaranteed in the sense that the transformed states, z, remain bounded inside the balls specified by the theorem. Consequently, the original physical states, x, remain bounded in ellipsoids determined by the diffeomorphism between x and z. Robust performance problem, however, is more involved than robust stabilization, since it requires judicious selection of the closed regions in z-domain that map to desired performance regions (that is, the ellipsoids) in the original x-domain.

Nonlinear systems after transformations (with and without uncertainty), and their stability results are summarized in Table 1. Next, the design procedure for matched and unmatched uncertainty will be demonstrated on a reactor system.

Application

Consider the isothermal reactor presented by Kravaris and Palanki (1988). The reaction $A \leftrightarrow B \rightarrow C$ takes place in a CSTR. The output is the concentration of C, and the control input is the molar feed flow rate of B. The dimensionless model is given by:

$$\dot{x}_1 = 1 - x_1 Da_1 x_1 + Da_2 x_2^2$$
$$\dot{x}_2 = -x_2 + Da_1 x_1 - Da_2 x_2^2 - Da_3 x_2^2 + u$$
$$\dot{x}_3 = -x_3 + Da_3 x_2^2$$
$$y = x_3$$

The variables are defined in Table 2. We are interested in designing controllers that achieve nominal I/O linearization and guarantee stability when some of the parameters Da_i are uncertain.

The relative order was found to be $r = 2$ as long as Da_3 is not zero, which is physically the case ($Da_3 = 0$ does not belong to Ω_p). The first matching conditions are satisfied since $g = [0\ 1\ 0]^T$ and $\delta g = 0$. The *nominal* transformations (Eq. 3) are given by:

$$z_1 = x_3 - x_3^0$$
$$z_2 = -x_3 + Da_3^0 x_2^2$$
$$z_3 = x_1 - x_1^0$$

$$u = \frac{v - 2Da_3^0 x_2(-x_2 + Da_1^0 x_1 - Da_2^0 x_2^2 - Da_3^0 x_2^2) - x_3 + Da_3^0 x_2^2}{2Da_3^0 x_2}$$

Note that $z^{(1)} = [z_1, z_2]$ and $z^{(2)} = z_3$. Both are linearly independent if $Da_3 \neq 0$. Also $L_g z_3 = 0\ \forall\ p \in \Omega_p$, thus the regularity conditions (a) and (b) are also satisfied.

The nominal values for the parameters are given by:

$$Da_1^0 = 3; \quad Da_2^0 = 0.5; \quad Da_3^0 = 1$$

If the nominal value for the input is set to $u^0 = 1$, the reactor has a single nominal operating point:

$$(x_1^0, x_2^0, x_3^0) = (0.3467, 0.8796, 0.7737)$$

Analysis

Applying the above nominal transformations, the NLS is mapped to the form given by Eq. 10:

Quasi-linear part:

$$\begin{bmatrix} \dot{z}_1 \\ \dot{z}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \dot{z}_1 \\ \dot{z}_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} v + \underline{L_{(\delta f + x)} z^{(1)}}$$

$$L_{(\delta f + x)} z^{(1)} = \begin{bmatrix} \dfrac{(Da_3 - Da_3^0)}{Da_3^0}(z_2 + z_1 + x_3^0) \\ 2Da_3^0 \sqrt{\dfrac{(z_2+z_1+x_3^0)}{Da_3^0}} \left[(Da_1 - Da_1^0)(z_3 + x_1^0) - \dfrac{(Da_2 - Da_2^0)}{Da_3^0}(z_2 + z_1 + x_3^0) - w \right] - w \end{bmatrix}$$

with $w = \dfrac{(Da_3 - Da_3^0)}{Da_3^0}(z_2 + z_1 + x_3^0)$

Perturbed zero dynamics part:

$$\dot{z}_3 = -(1 + Da_1^0) z_3 + \frac{Da_2^0}{Da_3^0}(z_1 + z_2)$$
$$\underline{-(\mathrm{Da}_1 - \mathrm{Da}_1^0)(z_3 + x_1^0) + \frac{(\mathrm{Da}_2 - \mathrm{Da}_2^0)}{\mathrm{Da}_3^0}(z_2 + z_1 + x_3^0)} \quad (12)$$

The underlined terms show the perturbations to the nominal system due to uncertainty.

Synthesis

First of all, assumptions 1-3 must be satisfied. The most critical one is assumption 3 dealing with the perturbed zero dynamics. We will identify (Eq. 12) with $\dot{x} = f[x(t), t, p]$ of theorem 1. It is easy to verify that this differential equation is also Lipschitz for all Da_1, Da_2, Da_3. If a stabilizing controller is to be implemented, the states of the quasi-linear part (z_1 and z_2) will be bounded (for all $t > 0$) by $\bar{r}$. Therefore, the following can be used for the second differential equation of theorem 1:

$$\dot{w} = -C_1 w + C_2 \quad \text{with} \quad C_1 = (1 + Da_1^0)$$

and

$$C_2 = \max_{\Omega_p} \left[2\bar{r} \left(\frac{Da_2^0}{Da_3^0} + \frac{(Da_2 - Da_2^0)}{Da_3^0} \right) - (\mathrm{Da}_1 - \mathrm{Da}_1^0) x_1^0 + \frac{(\mathrm{Da}_2 - \mathrm{Da}_2^0)}{\mathrm{Da}_3^0} x_3^0 \right]$$

so that $f(\cdot) \leq g(\cdot)$. It is easy to solve the second differential equation:

$$w(t) = \left(w(0) - \frac{C_2}{C_1} \right) e^{-C_1 t} + \frac{C_2}{C_1}$$

Since the initial condition $z_3(0)$ must be bounded by $\bar{r}$ (see theorem 2), if:

$$\frac{C_2}{C_1} < \bar{r} \quad (13)$$

then $z_3(t) \leq \bar{r}$ for every $t > 0$. Thus, when Eq. 13 holds, assumption 3 is satisfied.

The inequality (Eq. 13) gives an explicit relation between the magnitude of uncertainty (through C2) and the radius of the ball $B(\bar{r})$. For a value of $\bar{r}$ specified by the designer, Eq. 13 places a constraint on the magnitude of uncertainty allowed.

Assumption 1 vanishes for this example since $g = [0\ 1\ 0]^T$ and $\delta g = 0$. Assumption 2 depends on the uncertainty considered and will be checked when designing controllers for the different cases described below.

Case 1: uncertainty in Da_2. Consider a maximum of 10% uncertainty in Da_2:

$$\Omega_p = \{ Da_2 = Da_2^0 \pm \delta_i 0.1 Da_2^0; \quad 0 \leq \delta_i \leq 1 \}$$

For this uncertainty, the condition (Eq. 13) requires $\bar{r} > 0.035$. Therefore, we specify $\bar{r} = 0.04$. The least conservative ρ and μ are computed so that assumption 2 holds in the ball $B(\bar{r})$:

$$(\rho, \mu) = (0.072, 0.632)$$

Next, the PI controller gains are computed. Since $L_{\delta f + x} z^{(1)}$ is matched, we solve ARE for $H = I$ and $R = 1$. The gain and the δ^* plots are given in Figure 2. We choose $\delta^* = 0.039$ which has to be less than $\bar{r} = 0.04$ according to theorem 2. This value of δ^* corresponds to $\epsilon = 0.0094$ which fixes the gains as (see Figure 2):

$$(K_1, K_2, K_3) = (19.38, 12.31, 10.62)$$

where K_1, K_2, K_3 are proportional and integral gains, respectively. According to theorem 2, this controller will guarantee that the quasi-linear states z_1 and z_2 are δ-stable in a ball $B(\delta)$ ($\delta < 0.04$), and the zero dynamics state z_3 is bounded in $B(\delta = 0.04)$ for the following perturbations from nominal conditions:

- For all initial conditions of the quasi-linear states such that

$$\| z^{(1)}(t_o) \| < \frac{r}{\sqrt{33}}$$

for $r < 0.04$.

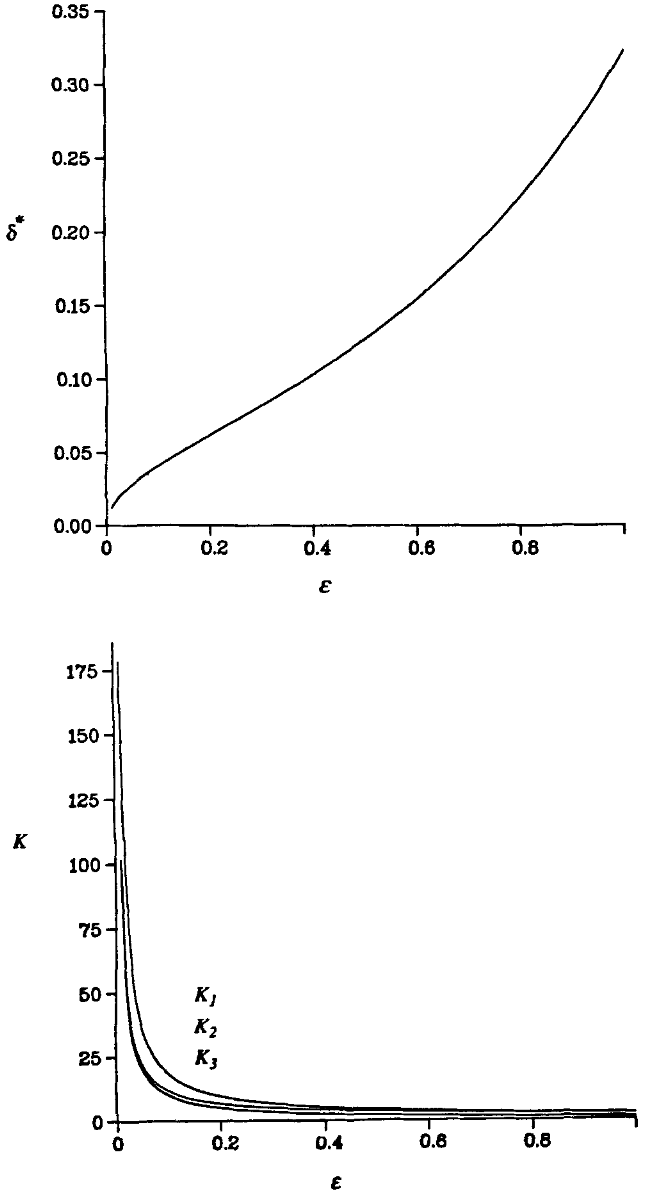

Figure 2. K and δ^* curves for case 1.

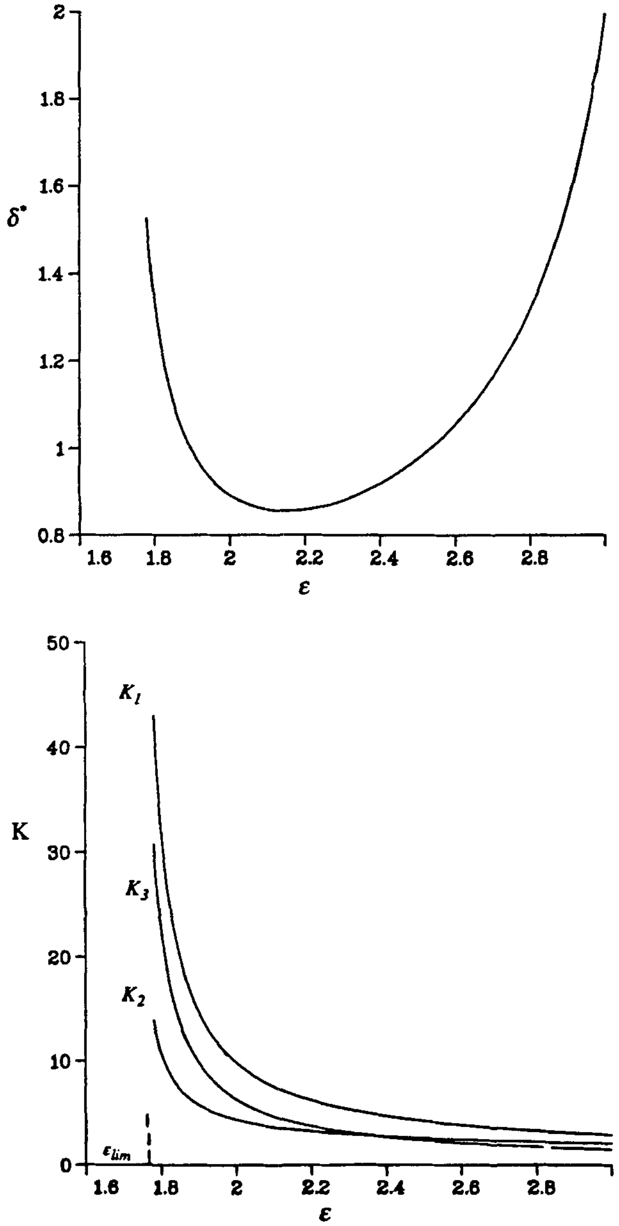

Figure 3. K and δ^* curves for case 2.

- For all initial conditions of the perturbed zero dynamics state z_3 such that $\|z_3(t_0)\| < 0.04$.
- For all uncertainty Ω_p.

Because of the one-to-one mapping between x and z, the states of the original NLS will also be bounded. In fact, the ball $B(\delta)$ within the state $z(t)$ lies is mapped to another closed contour in the x-state space within which $x(t)$ lies. It is also guaranteed that the output will exhibit zero steady-state offset.

Note that if the system is initially at steady state, the only perturbation is due to uncertainty, and robust stabilization is guaranteed in this case for 10% variations in Da_2. Finally, the region of guaranteed stability (the value δ) can be reduced, and the magnitude of uncertainty can be increased but both at the expense of higher controller gains. No matter how large the uncertainty is, the method will give stabilizing gains.

Case 2: uncertainty in Da_3. This uncertainty is considered to illustrate the fundamental difficulties associated with unmatched uncertainty. Assumption 3 is not required since the zero dynamics are not perturbed by Da_3 (Eq. 12). For 2% uncertainty, the bounds are given by:

$$(\rho, \mu) = (0.05, 0.316)$$

Since Da_3 induces unmatched uncertainty, one has to solve ARE′. For $H = I$ and $R = 0.1$, the gain plots are shown in Figure 3. Note that there is no solution for $\epsilon < \epsilon_{\lim} = 1.77$. The δ^* plot has a minimum at $\delta^*_{\min} = 0.85$. This means that δ-stability cannot be guaranteed for $\delta < 0.85$. Therefore, the value of $\delta = 0.86$ was chosen which yields the following PI gains:

$$(K_1, K_2, K_3) = (5.94, 3.13, 3.53)$$

For this uncertainty, δ-stability in an arbitrarily small neigh-

Table 2. Nomenclature for the Reactor

$$Da_1=\frac{k_1V}{F} \qquad x_1=\frac{C_A}{C_{A_f}}$$

$$Da_2=\frac{k_2VC_{A_f}}{F} \qquad x_2=\frac{C_B}{C_{A_f}}$$

$$Da_3=\frac{k_3VC_{A_f}}{F} \qquad x_3=\frac{C_C}{C_{A_f}}$$

$$\mu_1=\frac{N_{B_f}}{FC_{A_f}} \qquad t=t'\,\frac{F_0}{V}$$

borhood of the origin cannot be insured. Furthermore, the level of uncertainty cannot be arbitrarily increased since this requires smaller ϵ and larger K values which cannot be computed from ARE′. For example, for 5% uncertainty, for the smallest region of stability $B(\delta^*_{\min})$, one gets the following gains:

$$(K_1, K_2, K_3)=(13.2, 5.2, 9.3)$$

and for 10% uncertainty, the stabilizing gains cannot be computed.

Other cases. All the possible uncertainty combinations are shown in Table 3 together with the different stability results. Dynamic simulations confirm these results (Calvet, 1989).

Conclusion

The problem of robust stabilization of feedback, linearizable, uncertain and perturbed, nonlinear systems is tackled. The class of nonlinear systems are those which nominally admit input/output linearization. We consider bounded parametric uncertainty and external disturbances, and develop their structural effects on the transformed nonlinear system. A robust controller synthesis methodology guarantees the stability of the nonlinear system in a neighborhood of the nominal operating point. Uncertainty both with and without matching conditions are considered, and their ramifications are clarified. A reactor example used illustrates the new concepts and the procedure for robust controller design.

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Table 3. Different Stability Guarantees for the Reactor Application

	Uncertainties	$L_{\delta f}z^{(1)}$ Structured?	Z.D. Perturbed?	Kind of Stability with P-Control	Kind of Stability with PI-Control
1	Da_1	Yes (ARE)	Yes	δ-stabilization A.S. as $K\to\infty$	$\lim_{t\to\infty}\|z^{(1)}(t)\|=0$ $\|z^{(2)}(t)\|<\bar{r}\forall t>0$
1	Da_2	Yes (ARE)	Yes	δ-stabilization A.S. as $K\to\infty$	$\lim_{t\to\infty}\|z^{(1)}(t)\|=0$ $\|z^{(2)}(t)\|<\bar{r}\forall t>0$
1	Da_3	No (ARE′)	No	δ-stabilization $\delta^*_{\min}<\delta<\bar{r}$	$\|z^{(1)}(t)\|<\delta\forall t>T$ $\lim_{t\to\infty}\|z^{(2)}(t)\|=0$
2	Da_1, Da_2	Yes (ARE)	Yes	δ-stabilization A.S. as $K\to\infty$	$\lim_{t\to\infty}\|z^{(1)}(t)\|=0$ $\|z^{(2)}(t)\|<\bar{r}\forall t>0$
2	Da_1, Da_3	No (ARE′)	Yes	δ-stabilization $\delta^*_{\min}<\delta<\bar{r}$	$\|z^{(1)}(t)\|<\delta\forall t>T$ $\|z^{(2)}(t)\|<\bar{r}\forall t>0$
2	Da_2, Da_3	No (ARE′)	Yes	δ-stabilization $\delta^*_{\min}<\delta<\bar{r}$	$\|z^{(1)}(t)\|<\delta\forall t>T$ $\|z^{(2)}(t)\|<\bar{r}\forall t>0$
3	Da_1, Da_2, Da_3	No (ARE′)	Yes	δ-stabilization $\delta^*_{\min}<\delta<\bar{r}$	$\|z^{(1)}(t)\|<\delta\forall t>T$ $\|z^{(2)}(t)\|<\bar{r}\forall t>0$

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Appendix

Derivation of Eqs. 6 and 7

1. The nominal transformations are given by Eq. 3. By using the chain rule, the dynamics of the first $r-1$ states become:

$$\dot{z}_i=\frac{\partial(L_f^{i-1}h(x))}{\partial t}=\frac{\partial(L_f^{i-1}h(x))}{\partial x}\frac{\partial x}{\partial t}+\frac{\partial(L_f^{i-1}h(x))}{\partial p}\frac{\partial p}{\partial t}$$

Since the parameters are time invariant, the last term is zero and:

$$\dot{z}_i=\langle dL_f^{i-1}h(x), f+gu+\chi\rangle$$

$$\dot{z}_i=\langle dL_f^{i-1}h(x),\delta f+\hat{f}+\chi\rangle+\langle dL_f^{i-1}h(x),g\rangle u$$

$$=\langle dL_f^{i-1}h(x),\delta f+\hat{f}+\chi\rangle+\langle dL_f^{i-1}h(x),\hat{g}\rangle u + E\langle dL_f^{i-1}h(x),\hat{g}\rangle u$$

In the last equality, the first matching condition was used. Since the relative order is equal to r, the last two terms are zero; thus,

$$\dot{z}_i=L_f^ih+L_{(\delta f+\chi)}L_f^{i-1}h=z_{i+1}+L_{(\delta f+\chi)}z_i \quad \forall i=1,\ r-1$$

2. The dynamic of z_r is different and given by:

$$\begin{aligned}\dot{z}_r&=\langle dL_f^{r-1}h(x),\ f+gu+\chi\rangle\\&=\langle dL_f^{r-1}h,\ f\rangle+\langle dL_f^{r-1}h,\delta f+\chi\rangle+\langle dL_f^{r-1}h,\hat{g}\rangle u\\&\quad+\langle dL_f^{r-1}h,\delta g\rangle u\\&=L_f^rh+L_{(\delta f+\chi)}L_f^{r-1}h+L_{\hat{g}}L_f^{r-1}hu+L_{\delta g}L_f^{r-1}hu\end{aligned}$$

Substituting the nominal input transformation (Eq. 3):

$$\begin{aligned}\dot{z}_r&=v+L_{(\delta f+\chi)}L_f^{r-1}h+L_{\delta g}L_f^{r-1}hu\\&=v+L_{(\delta f+\chi)}L_f^{r-1}h+L_{\delta g}L_f^{r-1}h(\hat{\alpha}+\hat{\beta}v)\\&=(1+L_{\delta g}L_f^{r-1}h\hat{\beta})v+L_{(\delta f+\chi)}L_f^{r-1}h+L_{\delta g}L_f^{r-1}h\hat{\alpha}\\&=(1+L_{\delta g}z_r\hat{\beta})v+L_{(\delta f+\chi)}z_r+L_{\delta g}z_r\hat{\alpha}\end{aligned}$$

3. The zero dynamics are given by:

$$\begin{aligned}\dot{z}_j&=\frac{\partial z_j}{\partial t}=\frac{\partial z_j}{\partial x}(f+gu+\chi)=L_{f+\chi}z_j+L_gz_ju \quad j=r+1,\ldots,n\\&=L_{\hat{f}}z_j+L_{(\delta f+\chi)}z_j+L_{\hat{g}}z_ju+L_{\delta g}z_ju\\&=L_{\hat{f}}z_j+L_{(\delta f+\chi)}z_j+L_{\hat{g}}z_ju+EL_{\hat{g}}z_ju\end{aligned}$$

Due to regularity condition (b), the last two terms are zero:

$$\dot{z}_j=\hat{\xi}_{j-r}(z^{(1)},z^{(2)})+L_{(\delta f+\chi)}z_j \quad j=r+1,\ldots,n$$

Thus, the transformed NLS is given by:

$$\begin{aligned}\dot{z}^{(1)}&=Az^{(1)}+b(v+\eta(z,\delta g(z;p),v)+L_{(\delta f+\chi)}z^{(1)}\\\dot{z}^{(2)}&=\hat{\xi}(z^{(1)},z^{(2)})+L_{(\delta f+\chi)}z^{(2)}\\y&=cz^{(1)}=z_1\end{aligned}$$

Where (A, b) is the r-dimensional single-input BCF, and the nonlinear perturbations due to uncertainty are as defined in Eq. 7.

Proof of theorem 2

Here, we consider matched uncertainty only and a sketch is given. For complete proof, see Calvet (1989). Consider a Lyapunov function $V(z^{(1)})=(z^{(1)})^TPz^{(1)}$ and the control law $v(t)=-R^{-1}b^TPz^{(1)}(t)$. Under this control law the quasi-linear part (Eq. 11) becomes:

$$\dot{z}^{(1)}=Az^{(1)}+b[-R^{-1}b^TPz^{(1)}+\phi(z,p,d)-\psi(z,p)R^{-1}b^TPz^{(1)}]$$

Then

$$\frac{dV}{dt}=2z^{(1)^T}P\dot{z}^{(1)}=2z^{(1)^T}P[Az^{(1)}+b(-R^{-1}b^TPz^{(1)}+\phi(z,p,d) -\psi(z,p)R^{-1}b^TPz^{(1)})]$$

But

$$2z^{(1)T}PAz^{(1)}=z^{(1)T}(PA+AP)z^{(1)}$$

and inside $B(\bar{r})$ one has:

$$\begin{aligned}2z^{(1)^T}Pb\phi(z,p,d)&\le 2\|b^TPz^{(1)}\|\ \|\phi(z,p,d)\|\\&\le\|b^TPz^{(1)}\|^2+\|\phi(z,p,d)\|^2\\&\le z^{(1)^T}Pbb^TPz^{(1)}+\rho^2+\mu^2\|z^{(1)}\|^2\end{aligned}$$

Assumption 2 was used in the last inequality. Finally using assumption 1 it follows that:

$$\begin{aligned}-2z^{(1)^T}Pb\psi(z,p)R^{-1}b^TPz^{(1)}&\le 2\|b^TPz^{(1)}\|\ \|\psi(z,p)R^{-1}b^TPz^{(1)}\|\\&\le 2\bar{\alpha}\|b^TPz^{(1)}\|^2\|R^{-1}\|\\&\le 2\bar{\alpha}z^{(1)^T}Pbb^TPz^{(1)}\|R^{-1}\|\end{aligned}$$

Collecting all the terms:

$$\frac{dV}{dt} \leq z^{(1)^T}[PA + A^T P - Pb(2R^{-1} - (2\bar{\alpha}\|R^{-1}\| + 1)I)b^T P]z^{(1)} + \rho^2 + \mu^2 \|z^{(1)}\|^2$$

where the term inside the bracket shows the origin of the Riccati equation (ARE). Hence, from the algebraic Riccati equation it follows:

$$\frac{dV}{dt} \leq z^{(1)^T} \frac{H}{\epsilon^2} z^{(1)} + \rho^2 + \mu^2 \|z^{(1)}\|^2$$

Using the Rayleigh principle:

$$\leq -\frac{1}{\epsilon^2} \lambda_* (H) \|z^{(1)}\|^2 + \rho^2 + \mu^2 \|z^{(1)}\|^2$$

$$\leq -\left[\frac{\lambda_* (H) - \mu^2 \epsilon^2}{\epsilon^2}\right]\left[\|z^{(1)}\|^2 - \frac{\rho^2 \epsilon^2}{\lambda_* (H) - \mu^2 \epsilon^2}\right]$$

From here the completion of the proof can be done using a claim from Schmitendorf (1988b) as detailed by Calvet (1989).

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